The Effect of Lift on the Decay of a Circular Orbit†

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In the outer fringes of the atmosphere, the average rate of descent of a lifting vehicle is known to be a function of drag acceleration alone. Nearer the surface, the rate of descent depends on the lift-to-drag ratio. In this note, the transition region is examined and it is shown that lift becomes important when the dimensionless parameter $(L/mg)\beta r$ is large compared with unity.

The derivation that follows is similar to that of Hayes and Vander Velde.[1] They, however, were not concerned with a lifting vehicle.

The variable that is chosen to describe a trajectory at near-orbital velocity is flight-path angle. Writing the equation of motion in a direction perpendicular to the velocity vector in the form

$$V\frac{d\gamma}{dt} = g\left(1 - \frac{V^2}{V_c^2}\right) - \frac{L}{m} \tag{1}$$

it can be seen that the rate of change of flight-path angle is dependent on lift and the difference in velocity from local circular orbital velocity, V_c. To introduce this velocity difference into Eq. (1), make the substitution

$$V = V_c + \Delta V \tag{2}$$

Eq. (1) becomes

$$V_c\left(1 + \frac{\Delta V}{V_c}\right)\frac{d\gamma}{dt} = -\frac{2V_c}{r}\Delta V - \frac{1}{2}\rho V_c^2 \times \left(1 + \frac{2\Delta V}{V_c}\right)\frac{C_L A}{m} \tag{3}$$

where the term involving ΔV^2 has been neglected.

If the velocity is always close to local circular orbital velocity, $\Delta V/V_c$ can be neglected compared with unity, so that Eq. (3) becomes

$$\frac{d\gamma}{dt} = -\frac{2}{r}\Delta V - \frac{1}{2}\rho V_c\frac{C_L A}{m} \tag{4}$$

A second equation involving ΔV is obtained from the equation of motion in a direction parallel to the velocity vector:

$$dV/dt = -(D/m) + g\gamma \tag{5}$$

If Eq. (2) is substituted into Eq. (5) and use is made of the relation

$$dV_c/dt = {}^1/_2 g\gamma \tag{6}$$

one obtains

$$d\Delta V/dt = -(D/m) + {}^1/_2 g\gamma \tag{7}$$

To eliminate ΔV from Eqs. (4) and (7), take the derivative with respect to time of Eq. (4):

$$\frac{2}{r}\frac{d\Delta V}{dt} = -\frac{d^2\gamma}{dt^2} - \frac{1}{2}V_c\frac{C_L A}{m}\frac{d\rho}{dt} - \frac{1}{2}\rho\frac{C_L A}{m}\frac{dV_c}{dt} \tag{8}$$

Density is changing because of change in altitude. Assuming an exponential atmosphere, the density derivative is

$$\frac{d\rho}{dt} = -\beta\rho\frac{dr}{dt} = \beta\rho V_c\gamma \tag{9}$$

where β is the exponential-decay constant of the atmosphere. Substituting Eqs. (6) and (9) into Eq. (8) gives

$$2\frac{d\Delta V}{dt} = -r\frac{d^2\gamma}{dt^2} - \frac{L}{m}\left(\beta r + \frac{1}{2}\right)\gamma \tag{10}$$

For the earth, $\beta r \approx 900$, so that the 1/2 can be dropped. Substituting Eq. (7) into Eq. (10), one obtains finally

$$\frac{r}{g}\frac{d^2\gamma}{dt^2} + \left(1 + \frac{L}{mg}\beta r\right)\gamma = 2\frac{D}{mg} \tag{11}$$

The differential equation for flight-path angle is a linear, nonhomogeneous equation with variable coefficients. If the altitude change is small during one orbit, the coefficients can be regarded as constants. The solution of Eq. (11) is then composed of a sinusoidal oscillation and a steady-state term associated with the forcing function, $2(D/mg)$. An approximate value for the steady-state term is obtained by neglecting the second derivative in Eq. (11) and solving for γ. The result is

$$\gamma_{ss} = 2(D/mg)/[1 + (L/mg)\beta r] \tag{12}$$

γ_{ss} is proportional to the descent rate. Let us examine it for values of $(L/mg)\beta r$ large and small compared with unity.

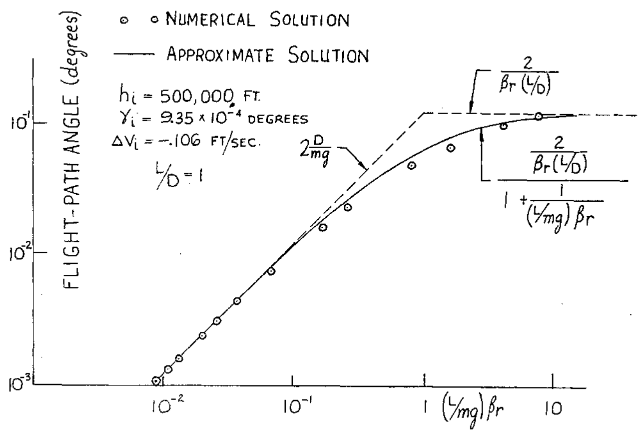

Fig. 1. Flight-path angle as a function of $(L/mg)\beta r$.

(1) $(L/mg)\beta r \gg 1$: If the 1 in the denominator of Eq. (12) is neglected, the flight-path angle is

$$\gamma_{ss} = (2/\beta r)(D/L) \tag{13}$$

This is the equilibrium-glide flight-path angle for velocities close to local circular orbital velocity.

(2) $(L/mg)\beta r \ll 1$: In this case, the flight-path angle is just $2(D/mg)$. This is the value for a nonlifting vehicle.

Thus, it is seen that lift is not important in describing the decay of an orbit if $(L/mg)\beta r$ is small compared with unity. Drag is the only significant aerodynamic force.

A check of this conclusion was made by numerically solving Eqs. (1) and (5). Initial conditions were selected so that the sinusoidal oscillation in flight-path angle would be suppressed. Thus, γ_i was set equal to the value given by Eq. (12), and ΔV was selected so that $d\gamma_i/dt$, given by Eq. (4), would be zero. The results are shown in Fig. 1, where flight-path angle is plotted as a function of $(L/mg)\beta r$. The numerical solution is seen to be quite close to the approximate solution of Eq. (12).

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Reference

[1] Hayes, J. E., and Vander Velde, W. E., *Satellite Landing Control System Using Drag Modulation*, ARS Journal, Vol. 32, No. 4, May 1962.